DEL PEZZO SURFACES OF DEGREE 6 OVER AN ARBITRARY FIELD

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ABSTRACT. We give a characterization of all del Pezzo surfaces of degree 6 over an arbitrary field F. A surface is determined by a pair of separable algebras. These algebras are used to compute the Quillen K-theory of the surface. As a consequence, we obtain an index reduction formula for the function field of the surface.

1. INTRODUCTION

If X is an algebraic variety defined over an arbitrary field F, a common method (cf. the introduction of [13]) for learning various properties of X is to first study $\overline{X} := X \times_{\mathrm{Spec}\, F} \mathrm{Spec}(\overline{F})$, the extension of scalars of X to a separable closure $\overline{F}$ of F, and then to study the action of the Galois group $\mathrm{Gal}(\overline{F}/F)$ on algebraic groups and other algebraic objects associated to $\overline{X}$. This is particularly useful when dealing with a class of varieties that all become isomorphic over $\overline{F}$, e.g. Severi-Brauer varieties or involution varieties. A Severi-Brauer variety is determined by a central simple F-algebra A, and an involution variety is determined by a central simple F-algebra A with an orthogonal involution of the first kind (A, σ). In either case this algebraic data determines geometrical and topological information about the corresponding variety. In particular the Quillen K-groups of the variety are determined the algebra in the Severi-Brauer example, and the algebra with involution in the involution variety example. This was proved for Severi-Brauer varieties and involution varieties in [8] and [11], respectively. Panin proved in [7] a more general theorem computing the K-theory of projective homogeneous varieties, which contains both examples as special cases. In all of these examples, as in this paper, the action of algebraic groups plays a significant role. An immediate consequence of this computation of the K-theory is an index reduction formula, which determines how extending scalars of a division F-algebra to the function field of the variety reduces the index of the algebra. In this paper we will study del Pezzo surfaces of degree 6 over F, obtaining similar results.

A del Pezzo surface S is a smooth projective surface over a field F such that the anti-canonical bundle ω_S^{-1} is ample. The degree (the self-intersection number of ω_S) of any such surface can be any integer between 1 and 9. Such varieties were discussed in [1], [2], and [12]. As mentioned in some of these references, a del Pezzo surface of degree 6 is a toric variety for a particular two dimensional torus, which we will describe below. We explore this toric structure in Section 2. The result is Theorem 2.4, a classification of all such surfaces

up to isomorphism preserving the action of the torus. Section 3 contains the main result of the paper, Theorem 3.5, where it is proved that a del Pezzo surface of degree 6 is determined by a pair B and Q of separable F-algebras, with centers K and L étale quadratic and cubic over F respectively, and both containing $K \otimes_F L$ as a subalgebra. Moreover, $\mathrm{cor}_{K/F}(B)$ and $\mathrm{cor}_{L/F}(Q)$ must be split. As an immediate corollary of Theorems 2.4 and 3.5, we give a necessary and sufficient condition in terms of B and Q for determining when the corresponding surface will have a rational point.

In Section 4, we relate the algebras B and Q to the endomorphism rings of locally free sheaves on the associated del Pezzo surface S. These sheaves are used in Theorem 4.2 to relate the Quillen K-theory of S to that of B and Q, by showing that the algebra $A = F \times B \times Q$ is isomorphic to S in a certain K-motivic category $\mathcal{C}$, constructed in [7]. This implies that for all n,

$$K_n(S) \cong K_n(A) = K_n(F) \oplus K_n(B) \oplus K_n(Q).$$

As a corollary we obtain an index reduction formula for the function field of S.

I would like to thank my advisor Alexander Merkurjev, who posed this question to me, and answered several of my questions which developed along the way.

We use the following notations and conventions:

An F-variety is a separated scheme of finite type over $\mathrm{Spec}(F)$.

$\overline{F}$ will denote a separable closure of F.

An F-algebra A is separable if $A \otimes_F L$ is semisimple for every field extension L of F. Such an algebra is Azumaya over its center, which is an étale extension of F.

Γ will denote the group $\mathrm{Gal}(\overline{F}/F)$.

For any F-variety X and any field extension E of F, we will denote $X \times_{\mathrm{Spec}\,F} \mathrm{Spec}(E)$ (resp. $X \times_{\mathrm{Spec}\,F} \mathrm{Spec}(\overline{F})$) by X_E (resp. $\overline{X}$).

For any separable F-algebra A and any étale extension E of F, we will denote $A \otimes_F E$ (resp. $A \otimes_F \overline{F}$) by A_E (resp. $\overline{A}$).

If D is a Cartier divisor on a variety X, $\mathcal{L}(D)$ will denote the corresponding invertible sheaf on X.

For any variety X and any separable algebra A, $\mathbf{P}(X; A)$ will denote the exact category of left $A \otimes_F \mathcal{O}_X$-modules which are locally free $\mathcal{O}_X$-modules. We will denote $\mathbf{P}(X; F)$ (resp. $\mathbf{P}(\mathrm{Spec}\,F; A)$) by $\mathbf{P}(X)$ (resp. $\mathbf{P}(A)$).

For any integer n, $K_n(X; A)$ will denote the Quillen group $K_n(\mathbf{P}(X; A))$. As above, we will denote $K_n(X; F)$ by $K_n(X)$ and $K_n(\mathrm{Spec}\,F; A)$ by $K_n(A)$.

For any algebraic torus T, $\widehat{T}$ will denote the Γ-module of characters $\mathrm{Hom}_{\overline{F}}(\overline{T}, \mathbb{G}_{m,\overline{F}})$.

2. TORIC VARIETIES

We first recall from [4], [5], and [12] some basic properties of the variety $\widetilde{S}$, the blow up of $\mathbb{P}^2$ at the 3 non-collinear points $[1 : 0 : 0]$, $[0 : 1 : 0]$, and $[0 : 0 : 1]$. The variety $\widetilde{S}$ can be realized as a closed subvariety of $\mathbb{P}^2 \times \mathbb{P}^2$, defined

by the equations $x_0 y_0 = x_1 y_1 = x_2 y_2$. The projection onto the first factor of $\mathbb{P}^2$ is the blow down of the three lines $m_0 = \{x_1 = x_2 = 0\}$, $m_1 = \{x_0 = x_2 = 0\}$, and $m_2 = \{x_0 = x_1 = 0\}$. Similarly, the projection onto the second factor of $\mathbb{P}^2$ is the blow down of the three lines $l_0 = \{y_1 = y_2 = 0\}$, $l_1 = \{y_0 = y_2 = 0\}$, and $l_2 = \{y_0 = y_1 = 0\}$.

Proposition 2.1. *Let $\widetilde{S}$ be the blow up of $\mathbb{P}^2$ at the three points $[1 : 0 : 0]$, $[0 : 1 : 0]$, and $[0 : 0 : 1]$.*

  i. *The variety $\widetilde{S}$ is a del Pezzo surface of degree 6 over F, and if F is separably closed, any del Pezzo surface S of degree 6 over F is isomorphic to $\widetilde{S}$.*
  ii. *The group $\mathrm{CH}^1(\widetilde{S})$ is generated by the lines l_0, l_1, l_2, m_0, m_1, and m_2.*
  iii. *The intersection pairing on $\mathrm{CH}^1(\widetilde{S})$ is determined by the following relations: $l_i^2 = -1$, $m_i^2 = -1$, $l_i m_j = 1$, and $l_i m_i = l_i l_j = m_i m_j = 0$, for distinct $i, j \in \{0, 1, 2\}$.*
  iv. *The group $\mathrm{CH}^2(\widetilde{S})$ is cyclic, generated by the class of any rational point.*

As mentioned in [1], there is an action of the torus $\widetilde{T} = \mathbb{G}_m^3 / \mathbb{G}_m$ on $\mathbb{P}^2$, described by:

$$(t_0, t_1, t_2) \cdot [x_0, x_1, x_2; y_0, y_1, y_2] = [t_0 x_0, t_1 x_1, t_2 x_2; t_0^{-1} y_0, t_1^{-1} y_1, t_2^{-1} y_2].$$

Here $\mathbb{G}_m$ embeds into $\mathbb{G}_m^3$ diagonally. This action sends $\widetilde{S}$ to itself, and is faithful and transitive on the open subset $\widetilde{U}$ of $\widetilde{S}$, the complement of the subvariety defined by the equation $x_0 x_1 x_2 y_0 y_1 y_2 = 0$. This closed subvariety has 6 irreducible components, the lines l_0, l_1, l_2, m_0, m_1, and m_2, which by the proposition are arranged in a hexagon. Thus $\widetilde{S}$ is a $\widetilde{T}$-toric variety, with fan dual to the hexagon of lines. There is also an action of the symmetric groups S_2 and S_3 on $\widetilde{S}$. The nontrivial element of S_2 acts on $\mathbb{P}^2 \times \mathbb{P}^2$ by interchanging the x_i and y_i, and the S_3 action on $\mathbb{P}^2 \times \mathbb{P}^2$ arises from the diagonal action of S_3 on the coordinates x_0, x_1, x_2 and y_0, y_1, y_2. The S_2 and S_3 actions commute with each other, and both groups send $\widetilde{S} \subset \mathbb{P}^2 \times \mathbb{P}^2$ to itself. Therefore they induce an action of $S_2 \times S_3$ on $\widetilde{S}$, preserving the set of lines l_0, l_1, l_2, m_0, m_1, and m_2, and thus inducing an isomorphism from $S_2 \times S_3$ onto the automorphism group of the hexagon of lines. The torus $\widetilde{T}$ is the connected component of the identity of the algebraic group $\mathrm{Aut}_F(\widetilde{S})$ of automorphisms of $\widetilde{S}$, and $S_2 \times S_3$ is the group of connected components. The action of $S_2 \times S_3$ on $\widetilde{S}$ define a section $S_2 \times S_3 \to \mathrm{Aut}_F(\widetilde{S})$, so we have the following split exact sequence of algebraic groups:

$$1 \to \widetilde{T} \to \mathrm{Aut}_F(\widetilde{S}) \to S_2 \times S_3 \to 1.$$
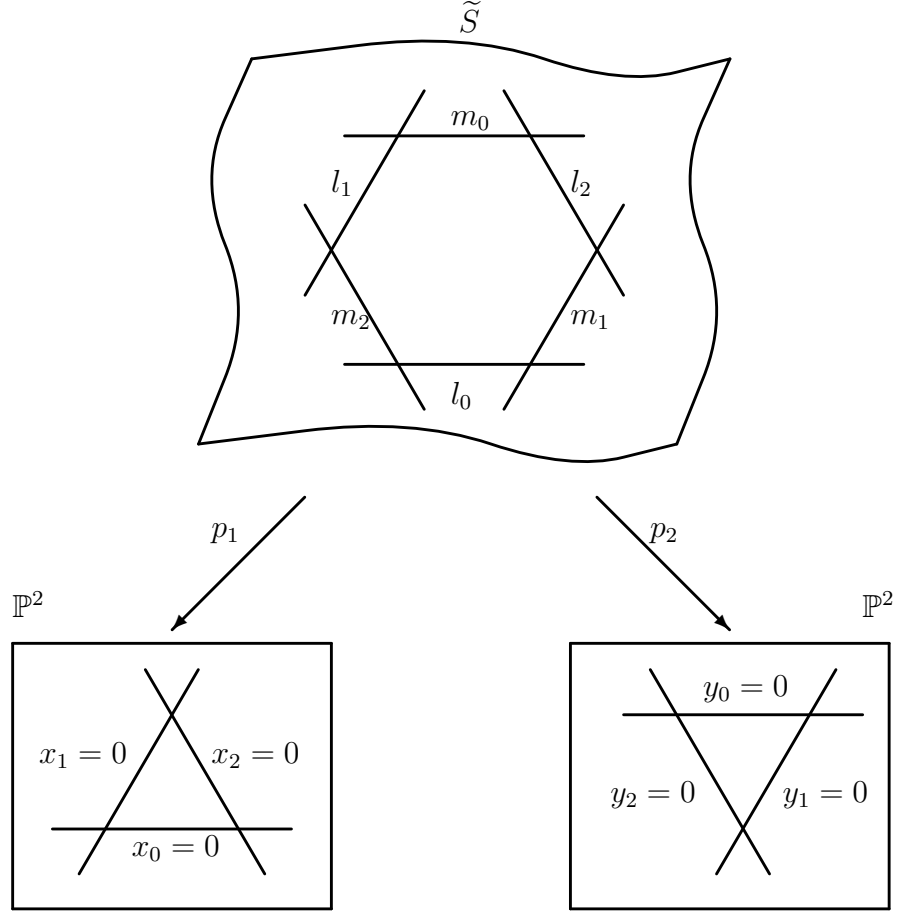
FIGURE 1. The Hexagon of Lines.

We have another way to realize $\widetilde{S}$ as a closed subvariety of a product of projective spaces. Define $f_i : \widetilde{S} :\to \mathbb{P}^1$ for $i = 0, 1, 2$ by

$$f_0([x_0 : x_1 : x_2; y_0 : y_1 : y_2]) = [x_1 : x_2] \;\; \text{or} \;\; [y_2 : y_1]$$
$$f_1([x_0 : x_1 : x_2; y_0 : y_1 : y_2]) = [x_2 : x_0] \;\; \text{or} \;\; [y_0 : y_2]$$
$$f_2([x_0 : x_1 : x_2; y_0 : y_1 : y_2]) = [x_0 : x_1] \;\; \text{or} \;\; [y_1 : y_0].$$

Each f_i is well defined, as the two definitions agree on the overlap, and thus is a morphism of varieties. These morphisms define a morphism $f : \widetilde{S} \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. If we denote the bi-homogeneous coordinates of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by X_0, X_1, Y_0, Y_1, Z_0, and Z_1, it can be shown that f maps $\widetilde{S}$ isomorphically onto the hypersurface of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined by the equation $X_0 Y_0 Z_0 = X_1 Y_1 Z_1$.

The morphism f sends $\widetilde{T}$ to the torus

$$\mathrm{Ker}(\mathbb{G}_m^2/\mathbb{G}_m \times \mathbb{G}_m^2/\mathbb{G}_m \times \mathbb{G}_m^2/\mathbb{G}_m \xrightarrow{m} \mathbb{G}_m^2/\mathbb{G}_m),$$

where $m((t_0, t_1), (t_0', t_1'), (t_0'', t_1'')) = (t_0 t_0' t_0'', t_1 t_1' t_1'')$.

Now let S be a del Pezzo surface of degree 6 over an arbitrary field F. Then $\overline{S}$ is a del Pezzo surface of degree 6 over $\overline{F}$, and thus by Proposition 2.1 is isomorphic over $\overline{F}$ to $\widetilde{\overline{S}}$. So S is an F-form of $\widetilde{S}$. As the six lines of the hexagon form a full set of exceptional curves in $\overline{S}$, the action of Γ on $\overline{S}$ is globally stable on the set of lines of the hexagon. Therefore, there is an open subvariety U whose complement Z is isomorphic over $\overline{F}$ to the hexagon of lines. The action of Γ on $\overline{Z}$ permutes its irreducible components, inducing an action of Γ on the hexagon.

Let T denote the connected component of the identity of $\mathrm{Aut}_F(S)$. The group of connected components G of $\mathrm{Aut}_F(S)$ is an étale group scheme: it is the group scheme determined (as in Proposition 20.16 of [3]) by the automorphism group of the hexagon of lines, with continuous Γ-action on this finite group as in the previous paragraph. So T is a torus, S is a T-toric variety, with an open set U which is a T-torsor, and Γ-action on the fan determined by the étale group scheme G.

This Γ-action on the hexagon determines a homomorphism $\gamma : \Gamma \to S_2 \times S_3$. Projecting onto either factor yields cocycles with values in S_2 and S_3, and thus γ determines a pair (K, L), where K and L are étale quadratic and cubic extensions of F, respectively. Note that while the fan, dual to the hexagon of lines, is the same for all del Pezzo surfaces of degree 6 over F, the possible Γ-actions on the fan are in a one-to-one correspondence with pairs (K, L).

For a fixed cocycle γ (i.e. a fixed pair (K, L)), we will classify all del Pezzo surfaces S of degree 6 where the Γ-action on $\overline{Z} \subset \overline{S}$ is determined by γ.

We have from [12] the following short exact sequence of Γ-modules:

$$0 \to \widehat{T} \to \mathbb{Z}[KL/F] \to \mathrm{Pic}(\overline{S}) \to 0.$$

Here KL denotes the algebra $K \otimes_F L$, and $\mathbb{Z}[KL/F]$ is the lattice of the six lines of $\overline{Z}$. The homomorphism $\mathbb{Z}[KL/F] \to \mathrm{Pic}(\overline{S})$ takes a line to the corresponding Cartier divisor on $\overline{S}$. As described in [1], this short exact sequence can be extended into the exact sequence

(1) $\qquad 0 \to \widehat{T} \to \mathbb{Z}[KL/F] \to \mathbb{Z}[K/F] \oplus \mathbb{Z}[L/F] \to \mathbb{Z} \to 0.$

Here $\mathbb{Z}[L/F]$ is the lattice of pairs of opposite lines, and $\mathbb{Z}[K/F]$ is the lattice of triangles, where each triangle is a triple of skew lines. The homomorphism $\mathbb{Z}[KL/F] \to \mathbb{Z}[L/F]$ sends each line to the pair containing it, and the homomorphism $\mathbb{Z}[KL/F] \to \mathbb{Z}[K/F]$ sends each line to the triangle containing it. The homomorphism $\mathbb{Z}[K/F] \oplus \mathbb{Z}[L/F] \to \mathbb{Z}$ is the difference of the augmentation maps. This sequence induces the following short exact sequence of Γ-modules:

(2) $\qquad 0 \to \widehat{T} \to \mathbb{Z}[KL/F]/\mathbb{Z} \to \mathbb{Z}[K/F]/\mathbb{Z} \oplus \mathbb{Z}[L/F]/\mathbb{Z} \to 0.$

where $\mathbb{Z}$ embeds into $\mathbb{Z}[K/F]$, $\mathbb{Z}[L/F]$, and $\mathbb{Z}[KL/F]$ diagonally.

In analogy with $R_{K/F}^{(1)}(\mathbb{G}_m) := \mathrm{Ker}(N_{K/F} : R_{K/F}(\mathbb{G}_m) \to \mathbb{G}_m)$, we define the following algebraic F-groups:

$$\mathbf{G}_L := \mathrm{Ker}(N_{KL/L} : R_{KL/F}(\mathbb{G}_m) \to R_{L/F}(\mathbb{G}_m))$$
$$\mathbf{G}_K := \mathrm{Ker}(N_{KL/K} : R_{KL/F}(\mathbb{G}_m) \to R_{K/F}(\mathbb{G}_m))$$

These groups are F-tori, dual to the Γ-modules $\mathbb{Z}[KL/F]/\mathbb{Z}[L/F]$ and $\mathbb{Z}[KL/F]/\mathbb{Z}[K/F]$, where $\mathbb{Z}[K/F]$ and $\mathbb{Z}[L/F]$ are diagonally embedded in $\mathbb{Z}[KL/F]$. The embeddings of $R_{K/F}(\mathbb{G}_m)$ and $R_{L/F}(\mathbb{G}_m)$ into $R_{KL/F}(\mathbb{G}_m)$ induce embeddings $R_{K/F}^{(1)}(\mathbb{G}_m) \to \mathbf{G}_L$ and $R_{L/F}^{(1)}(\mathbb{G}_m) \to \mathbf{G}_K$. The description of $\widetilde{T} \subset \widetilde{S} \subset \mathbb{P}^2 \times \mathbb{P}^2$ above descends to the following exact sequence:

$$(3) \qquad 1 \to R_{K/F}^{(1)}(\mathbb{G}_m) \to \mathbf{G}_L \to T \to 1.$$

Similarly, the description of $f(\widetilde{T}) \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ above descends to

$$1 \to R_{L/F}^{(1)}(\mathbb{G}_m) \to \mathbf{G}_K \to T \to 1.$$

We will use these sequences in Section 4.

Finally, from (1) and (2), we have corresponding sequences of F-tori:

$$(4) \qquad 1 \to \mathbb{G}_m \to R_{K/F}(\mathbb{G}_m) \times R_{L/F}(\mathbb{G}_m) \to R_{KL/F}(\mathbb{G}_m) \to T \to 1,$$

and

$$1 \to R_{K/F}^{(1)}(\mathbb{G}_m) \times R_{L/F}^{(1)}(\mathbb{G}_m) \xrightarrow{\phi} R_{KL/F}^{(1)}(\mathbb{G}_m) \to T \to 1.$$

Recall that for $E = K$, L, and KL,

$$H^1(F, R_{E/F}^{(1)}(\mathbb{G}_m)) = F^\times / N_{E/F}(E^\times)$$
$$H^2(F, R_{E/F}^{(1)}(\mathbb{G}_m)) = \mathrm{Ker}(\mathrm{cor}_{E/F} : \mathrm{Br}(E) \to \mathrm{Br}(F)).$$

Moreover, as $N_{KL/F}((KL)^\times)$ is a subgroup of $N_{K/F}(K^\times)$ and $N_{L/F}(L^\times)$, it follows that the restriction of the homomorphism of H^1 groups induced by ϕ to either factor is just factoring out the corresponding subgroup of the quotient, and thus ϕ will be surjective. Therefore, by the induced long exact sequence in cohomology, we obtain the following exact sequence:

$$1 \to H^1(F, T) \to \mathrm{Ker}(\mathrm{cor}_{K/F}) \times \mathrm{Ker}(\mathrm{cor}_{L/F}) \to \mathrm{Ker}(\mathrm{cor}_{KL/F}).$$

where the last homomorphism sends a pair (x, y) to $\mathrm{res}_{KL/K}(x) - \mathrm{res}_{KL/L}(y) \in \mathrm{Br}(KL)$.

Let C_1 be the set of K-algebra isomorphism classes of Azumaya K-algebras B of rank 9 such that $B_L = B \otimes_K KL$ and $\mathrm{cor}_{K/F}(B)$ are split, C_2 the set of L-algebra isomorphism classes of Azumaya L-algebras Q of rank 4 such that $Q_K = Q \otimes_L KL$ and $\mathrm{cor}_{L/F}(Q)$ are split, and set $C = C_1 \times C_2$. Then C is a pointed set with distinguished element $(M_3(K), M_2(L))$, and the map $\psi : C \to \mathrm{Ker}(\mathrm{cor}_{K/F}) \times \mathrm{Ker}(\mathrm{cor}_{L/F})$ sending a pair (B, Q) to $([B], [Q])$ is a morphism of pointed sets. Moreover, $\mathrm{res}_{KL/K}([B]) = [B \otimes_K KL]$ and $\mathrm{res}_{KL/L}([Q]) = [Q \otimes_L KL]$ are trivial, so it follows that ψ maps into $H^1(F, T)$.

Theorem 2.2. $\psi : C \to H^1(F,T)$ *is an isomorphism of pointed sets.*

Proof. If $\psi(B,Q) = \psi(B',Q')$, then $[B] = [B'] \in \mathrm{Ker}(\mathrm{cor}_{K/F}) \subset \mathrm{Br}(K)$. Then B and B' are similar Azumaya K-algebras of the same rank, and thus must be isomorphic as K-algebras. Similarly, Q and Q' are isomorphic, so that ψ is injective.

Now let $(x,y) \in H^1(F,T)$, so that $(x,y) \in \mathrm{Ker}(\mathrm{cor}_{K/F}) \times \mathrm{Ker}(\mathrm{cor}_{L/F})$, and $\mathrm{res}_{KL/K}(x) = \mathrm{res}_{KL/L}(y)$. This implies that

$$
\begin{aligned}
3x &= \mathrm{cor}_{KL/K}(\mathrm{res}_{KL/K}(x)) \\
&= \mathrm{cor}_{KL/K}(\mathrm{res}_{KL/L}(y)) = \mathrm{res}_{K/F}(\mathrm{cor}_{L/F}(y)) = 0.
\end{aligned}
$$

Similarly, $2y = 0$. Thus $\mathrm{res}_{KL/K}(x) = \mathrm{res}_{KL/L}(y)$ has order divisible by 2 and 3, and therefore is trivial. If L is not a field, then $L = F \times E$, where E is an étale quadratic extension of F. Then $\mathrm{res}_{KL/K}(x) = (x, \mathrm{res}_{E \otimes_F K/K}(x))$, and so $\mathrm{res}_{KL/K}(x) = 0$ implies $x = 0$. If L is a field, then x is split by a field extension of degree 3 (If $K = F \times F$, and $x = (x_1, x_2) \in \mathrm{Br}(K) = \mathrm{Br}(F) \times \mathrm{Br}(F)$ is split by KL if and only if x_1 and x_2 are split by L). Thus for all possible K and L, there is an Azumaya K-algebra B of rank 9 that represents x in $\mathrm{Br}(K)$. Since $\mathrm{res}_{KL/K}(x)$ and $\mathrm{cor}_{K/F}(x)$ are trivial, $B \otimes_K KL$ and $\mathrm{cor}_{K/F}(B)$ are split. Similarly, there is an Azumaya L-algebra Q of rank 4 which represents y such that $Q \otimes_L KL$ and $\mathrm{cor}_{L/F}(Q)$ are split. Then $(B,Q) \in A$, and $\phi(B,Q) = (x,y)$, so ψ is surjective. $\qquad\square$

Remark 2.3. As KL is an étale algebra of degree 3 over K, if KL splits B, then KL can be embedded as a subalgebra of B. Similarly, KL can be embedded as a subalgebra of Q. If $(B,Q) = (B',Q')$ in $C = H^1(F,T)$, then any K-isomorphism from B to B' sends $KL \subset B$ to a subalgebra of B' isomorphic to KL. Moreover, if we choose a fixed embedding of KL into both B and B', by applying Skolem-Noether to B' we can find an isomorphism from B to B' which restricts to the identity on KL. Similarly, we may assume that Q to Q' are isomorphic via an isomorphism which is the identity on KL.

It follows that if K and L are étale quadratic and cubic extensions of F respectively, and T is the two dimensional torus induced from K and L as in the exact sequence (4), then elements of $H^1(F,T)$ are determined by triples (B, Q, KL), where B is an Azumaya K-algebra of rank 9 such that $\mathrm{cor}_{K/F}(B)$ is split, Q is an Azumaya algebra over L of rank 4 such that $\mathrm{cor}_{L/F}(Q)$ is split, and we have a fixed embedding of KL as a subalgebra into both B and Q. Two triples (B, Q, KL) and (B', Q', KL) will determine the same element of $H^1(F,T)$ if there are KL-algebra isomorphisms from B to B' and Q to Q'.

If S is a del Pezzo surface of degree 6, and if T is the connected component of the identity of $\mathrm{Aut}_F(S)$, S is a T-toric variety, with Γ-action on the fan induced by the Γ-action γ on the connected components of $\overline{Z}$, the hexagon of lines. The T-torsors $U \subset S$ is determined by an element of the pointed set $H^1(F,T)$. Two surfaces S and S' will be isomorphic as toric varieties if and only if T and T' are isomorphic as algebraic groups, and there is an

isomorphism from S to S' which preserves the action of $T \cong T'$ on S and S', thus inducing isomorphisms Γ-actions on the fan and isomorphisms of the T-torsors determining S and S'. Thus we have proved the following:

Theorem 2.4. *Let S be a del Pezzo surface of degree 6, and T be the connected component of the identity of the group $\mathrm{Aut}_F(S)$. Then S is a T-toric variety with Γ-invariant fan determined by a pair (K, L) and the T-torsor U determined by a triple (B, Q, KL). Two triples (B, Q, KL) and $(B', Q', K'L')$ will describe isomorphic toric varieties if and only if K and L are isomorphic to K' and L' as F-algebras, (so that $T \cong T'$), and there exist KL-algebra isomorphisms from B to B' and Q to Q'.*

3. The Main Theorem

We would now like to classify these surfaces up to isomorphism as abstract varieties. This is less restrictive than isomorphism as toric varieties. We will see that a del Pezzo surface is still determined by a triple (B, Q, KL), but now we will allow F-algebra isomorphisms on B and Q, i.e. algebra isomorphisms which may not fix KL.

Let S be a del Pezzo surface of degree 6 over F. Then S is a T-toric variety, where T is the connected component of $\mathrm{Aut}_F(S)$, with the Γ-action on the fan determining a pair (K, L), and the T-torsor $U \subset S$ determining a pair (B, Q, KL). Let G be the group of connected components of $\mathrm{Aut}_F(S)$. As we have shown above, G is an étale group scheme, determined by the action of Γ on the hexagon of lines. In particular, $G(F) = \mathrm{Aut}_F(KL) \cong \mathrm{Aut}_F(K) \times \mathrm{Aut}_F(L)$.

Consider the following action of $G(F)$ on $H^1(F, T)$: if $(g, h) \in \mathrm{Aut}_L(KL) \times \mathrm{Aut}_K(KL)$ and $(B, Q, KL) \in H^1(F, T)$ then g nontrivial sends B to B^{op}, and sends the embedding $i : KL \to B$ to $i^g : KL \to B^{\mathrm{op}}$, where $i^g(z) = i(g(z))^{\mathrm{op}} \in B^{\mathrm{op}}$. As KL is a cyclic extension of L, Q is a cyclic L-algebra, so there is an element $l \in L^\times$ such that Q is generated by KL and an element y, subject to the relations $y^2 = l$ and $zy = y\sigma(z)$ for every $z \in KL$, where $\sigma \in \mathrm{Aut}_L(KL)$ is the nontrivial automorphism of KL over L. Such an algebra is denoted $(KL/L, l)$. Let h act on $Q = (KL/L, L)$ by $h \cdot (KL/L, l) = (KL/L, h(l))$, and send the embedding $KL \to Q$ to $KL \xrightarrow{h} KL \to (KL, h(l))$. As $l \in L^\times$ determines $Q = (KL/L, l)$ up to multiplication by an element of the subgroup $\mathrm{N}_{KL/L}\big((KL)^\times\big) \subset L^\times$, and as $\mathrm{Aut}_F(L)$ takes $\mathrm{N}_{KL/L}\big((KL)^\times\big)$ to itself, we see that this action is well defined.

The orbits of this group action can be described in terms of F-algebra isomorphisms on B and Q, as we will show below. We will use the following proposition several times: We will need the following proposition:

Proposition 3.1 (Proposition 4.18 of [3]). *Let (B, τ) be a central simple F-algebra of degree n with unitary involution, and let K be the center of B. For every F-subalgebra L of B which is étale of dimension n over F, there exists a unitary involution of B fixing L.*

Proposition 3.2. $(B', Q') = g \cdot (B, Q)$ *for some* $g \in G(F)$ *if and only if there are* F*-automorphisms* $\phi_B : B \to B'$ *and* $\phi_Q : Q \to Q'$ *such that* $\phi_B|_{KL} = \phi_Q|_{KL} = g$.

Proof. Assume that $(B', Q', KL) = g \cdot (B, Q, KL)$ for some $g \in G(F)$. Any F-automorphism g of KL can be expressed as the composition of two automorphisms, one fixing K and one fixing L. So it suffices to consider the separate cases where K and L are fixed by the automorphism.

We first consider the case where g fixes K, so that $B' = B$. By Skolem-Noether, there is a K-automorphism ϕ_B of B such that $\phi_B|_{KL} = g$. Now, $Q = (KL/L, l)$, $Q' = (KL/L, g(l))$ and as σ and g commute, g extends to an F-automorphism ϕ_Q from Q to Q', by sending KL to KL via g, and y to y'. Then ϕ_B and ϕ_Q agree on KL.

Now assume that $g = \sigma$ is the non-trivial L-automorphism of KL, so that $Q' = Q$ and $B' = B^{\mathrm{op}}$. As in the previous paragraph, by Skolem-Noether there is an L-automorphism ϕ_Q of Q such that $\phi_Q|_{KL} = g$. Moreover, as $\mathrm{cor}_{K/F}(B)$ is split and L is an étale cubic extension of F, we know by Proposition 3.1 that B has a unitary involution τ which is the identity on L. The involution τ defines an F-isomorphism ϕ_B from B to B^{op}, such that $\phi_B|_{KL} = \sigma = g = \phi_Q|_{KL}$.

Conversely, assume that $\phi_B : B \to B'$ and $\phi_Q : Q \to Q'$ are F-isomorphisms such that $\phi_B|_{KL} = \phi_Q|_{KL} = g \in \mathrm{Aut}_F(KL)$. As in the arguments above, we will first consider the separate cases where g fix K and L.

If g fixes L, then ϕ_Q is an isomorphism of L-algebras, so that $(B', Q, KL) = (B', Q', KL)$ in $H^1(F, T)$. If we restrict ϕ_B to the center of B, we get an F-isomorphism of K. If $\phi|_K$ is the identity (i.e. g is trivial), then B and B' are isomorphic as K-algebras. If $\phi|_K$ is not the identity, then by pre-composing ϕ with the isomorphism from B^{op} to B induced by any unitary involution τ fixing L on B, we see that B^{op} and B' are isomorphic as K-algebras. In either case, $(B', Q, KL) = (B', Q, KL) = g \cdot (B, Q, KL)$ in $H^1(F, T)$.

Now assume that g fixes K, so that $\phi_B : B \to B'$ is an isomorphism of K-algebras, and then $(B', Q', KL) = (B, Q', KL)$ in $H^1(F, T)$. If $Q = (KL/L, l)$, and if $l' = \phi_Q(l) = g(l)$, then Q is isomorphic over L to $(KL/L, l')$, and so $(B', Q', KL) = (B, Q', KL) = g \cdot (B, Q)$ in $H^1(F, T)$.

Finally, assume that g does not fix K or L. If σ is the nontrivial L-automorphism of KL, σg does fix K. Moreover, by post-composing ϕ_B with the F-isomorphism $\phi_{B'} : B' \to (B')^{\mathrm{op}}$ induced by any unitary involution τ of B' fixing L (which exist by Proposition 3.1), we get isomorphisms $\phi_{B'} \circ \phi_B : B \to (B')^{\mathrm{op}}$ and $\phi_Q : Q \to Q'$ such that $(\phi_{B'} \circ \phi_B)|_{KL} = \phi_Q|_{KL} = \sigma g$. Therefore by our argument in the previous paragraph, $\sigma \cdot (B', Q', KL) = ((B')^{\mathrm{op}}, Q', KL) = \sigma g \cdot (B, Q, KL)$ in $H^1(F, T)$. Acting on both sides of this equation by σ, we get $(B', Q', KL) = g \cdot (B, Q, KL)$. $\qquad \square$

The next theorem relates isomorphism classes of del Pezzo Surfaces of degree 6 with $G(F)$-orbits of $H^1(F, T)$. We will need the following standard result from Galois cohomology: (cf. Corollary (28.10) of [3] or Chapter I, Section 5.5, Corollary 2 of [9].)

Proposition 3.3. *Let Γ be a profinite group, A and B be Γ-groups with A a normal subgroup of B, and set $C = B/A$. If $\beta \in H^1(\Gamma, B)$, and b a cocycle representing β, then the elements of $H^1(\Gamma, B)$ with the same image as β in $H^1(\Gamma, C)$ corresponding bijectively with $(C_b)^\Gamma$-orbits of the set $H^1(\Gamma, A_b)$.*

Theorem 3.4. *The isomorphism class of S corresponds to a $G(F)$-orbit of $H^1(F, T)$.*

Proof. As mentioned in Section 2, we have the following split exact sequence of algebraic groups:

$$(5) \qquad\qquad 1 \to \widetilde{T} \to \mathrm{Aut}_F(\widetilde{S}) \to S_2 \times S_3 \to 1,$$

where $\widetilde{T}$ is the connected component of the identity of $\mathrm{Aut}_F(\widetilde{S})$, and $S_2 \times S_3$ is the group of connected components. This sequence induces the split exact sequence of pointed sets:

$$1 \to H^1(F, \widetilde{T}) \to H^1(F, \mathrm{Aut}_F(\widetilde{S})) \to H^1(F, S_2 \times S_3) \to 1.$$

The elements of $H^1(F, \mathrm{Aut}_F(\widetilde{S}))$ are in a one-to-one correspondence with the set of isomorphisms classes of F-forms of $\widetilde{S}$, which by Proposition 2.1 are del Pezzo surfaces of degree 6. Let $\beta \in Z^1(F, \mathrm{Aut}_F(\widetilde{S}))$ be a cocycle whose cohomology class is determined by the isomorphism class of S, and let γ be the image of β in $Z^1(F, S_2 \times S_3)$. The cocycle γ is determined by the action of the Galois group Γ on $\overline{Z} \subset \overline{S}$, and induces a pair (K, L). The twist of $\widetilde{T}$ by γ is the torus T, determined by (K, L) as in the sequence (4), and the twist of $S_2 \times S_3$ is the étale group scheme G. The result follows by Proposition 3.3. $\square$

Note that as $H^1(F, \mathrm{Aut}_F(\widetilde{S})) \to H^1(F, S_2 \times S_3)$ is surjective, we see that all possible pairs (K, L) are realized by the action of Γ on $\overline{Z}$, for Z contained in some del Pezzo surface S of degree 6. So let S_1 and S_2 be two del Pezzo surfaces of degree 6, and let $(B_i, Q_i, K_i L_i)$ be an element of the the $G_i(F)$-orbit of $H^1(F, T_i)$ determined by S_i, for $i = 1, 2$. Then S_1 and S_2 induce isomorphic Γ-actions on the hexagon of lines (so that $(K_1, L_1) \cong (K_2, L_2)$, $T_1 \cong T_2$, and $G_1 \cong G_2$; we denote these algebraic objects (K, L), T, and G, respectively). It follows from Proposition 3.2 that two pairs (B_1, Q_1, KL) and (B_2, Q_2, KL) are in the same $G(F)$-orbit of $H^1(F, T)$ if and only if there are isomorphisms $\phi_{B_1} : B_1 \to B_2$ and $\phi_{Q_1} : Q_1 \to Q_2$ such that $\phi_{B_1}|_{KL} = \phi_{Q_1}|_{KL}$. We have proved the following theorem:

Theorem 3.5. *There are bijections, inverse to each other, between the following two sets:*

- *The set of isomorphism classes of del Pezzo surfaces of degree 6.*
- *The set of triples (B, Q, KL), modulo the relation: $(B, Q, KL) \sim (B', Q', K'L')$ if there are F-algebra isomorphisms $\phi_B : B \to B'$ and $\phi_Q : Q \to Q'$ such that $\phi_B|_{KL} = \phi_Q|_{KL}$.*

For the rest of this paper, $S(B, Q, KL)$ will denote the del Pezzo surface of degree 6 determined by the triple (B, Q, KL).

Corollary 3.6. *The surface $S(B, Q, KL)$ contains a rational point if and only if B and Q are split.*

Proof. Since $S(B, Q, KL)$ is a T-toric variety for a two dimensional torus T, S has a rational point if and only if the corresponding T-torsor U is a trivial torsor (cf. Proposition 4 of [14]). By Theorem 2.4, this occurs precisely when B and Q are split. $\square$

Remark 3.7. If S_0 is a T-toric model (i.e. the T-torsor $U \subset S_0$ is trivial), then the map $H^1(F, T) \to H^1(F, \mathrm{Aut}_F(S))$ induced by $T \hookrightarrow \mathrm{Aut}_F(S)$ takes a T-torsor U to the surface determined by U and the Γ-action on the fan determined by the pair (K, L). Thus for any surface S, the elements of $H^1(F, T)$ in the fiber of the isomorphism class of S determine the possible non-isomorphic T-toric structures on S, where T is the connected component of the identity of the algebraic group $\mathrm{Aut}_F(S)$. In terms of the algebras B and Q, the map $H^1(F, T) \to H^1(F, \mathrm{Aut}_F(S))$ forgets the KL-algebra structure of B and Q, preserving only the F-algebra structure and the embedding of KL into B and Q.

The group $\mathrm{Aut}_F(K)$ always has order 2, but the group $\mathrm{Aut}_F(L)$ can have order 1, 2, 3, or 6. If $\mathrm{Aut}_F(L)$ has order less than 6, then the orbit of (B, Q, KL) in $H^1(F, T)$ contains at most 6 elements. It $\mathrm{Aut}_F(L)$ has order 6, then $L = F^3$ is not a field, and thus B is necessarily split. If B is split, then the pair $(B, Q, KL) \in H^1(F, T)$ is fixed by the subgroup $\mathrm{Aut}_F(K)$ of $G(F)$, and so again the $G(F)$-orbit of (B, Q, KL) in $H^1(F, T)$ has at most 6 elements. Thus for a del Pezzo surface S of degree 6, there at most 6 non-isomorphic T-toric structures on S.

Remark 3.8. We would like to relate this characterization of del Pezzo surfaces of degree 6 by triples (B, Q, KL) with the characterization by triples (B, τ, L) found in [1]. A triple (B, τ, L) is an Azumaya K-algebra B of rank 9, a unitary involution τ on B, and a cubic étale F-algebra L such that $L \subset \mathrm{Sym}(B, \tau)$. Two triples (B, τ, L) and (B', τ', L') are isomorphic if there is an F-algebra isomorphism $\phi : B \to B'$ such that $\tau'\phi = \phi\tau$ and $\phi(L) = L'$.

So let (B, Q, KL) be a triple as in Theorem 3.5. Then B is an Azumaya K-algebra of rank 9, and is classified up to isomorphism as an F-algebra. As B_L is split, so B contains KL, and hence L, as a subalgebra. Since $\mathrm{cor}_{K/F}(B)$ is split, we know that B has a unitary involution. Moreover, since L is an étale cubic extension of F contained in B, there is some unitary involution τ such that $L \subset \mathrm{Sym}(B, \tau)$, by Proposition 3.1. So the B and L in our characterization match with the B and L described in [1]. The rest of the remark seeks to relate Q and the involution τ. That is, we want to classify all triples (B, τ, L) with B and L fixed. This should correspond to fixing K, L, and B, and trying to determine all possible Q.

As K/F and KL/L are cyclic,

$$\mathrm{Br}(K/F) \cong F^\times / \mathrm{N_{K/F}}(\mathrm{K}^\times)$$

and

$$\mathrm{Br}(KL/L) \cong L^{\times}/\mathrm{N}_{\mathrm{KL/L}}\big((KL)^{\times}\big).$$

The restriction homomorphism $\mathrm{res}_{L/F} : \mathrm{Br}(F) \to \mathrm{Br}(L)$ sends the subgroup $\mathrm{Br}(K/F)$ to $\mathrm{Br}(KL/L)$. As K and L have coprime degrees, $\mathrm{res}_{L/F}\,|_{\mathrm{Br}(K/F)} :$ $\mathrm{Br}(K/F) \to \mathrm{Br}(KL/L)$ is injective, and the cyclic algebra Q corresponds to an element of $\mathrm{Br}(KL/L)/\mathrm{res}_{L/F}(\mathrm{Br}(K/F))$, i.e. an element of

$$L^{\times}/\mathrm{N}_{\mathrm{KL/L}}\big((KL)^{\times}\big) \Big/ \mathrm{res}_{L/F}\left(F^{\times}/\mathrm{N}_{\mathrm{K/F}}(K^{\times})\right)$$

$$= L^{\times}/\mathrm{N}_{\mathrm{KL/L}}\big((KL)^{\times}\big) \Big/ F^{\times}\,\mathrm{N}_{\mathrm{KL/L}}\big((KL)^{\times}\big)/\mathrm{N}_{\mathrm{KL/L}}\big((KL)^{\times}\big)$$

$$\cong L^{\times}/F^{\times}\,\mathrm{N}_{\mathrm{KL/L}}\big((KL)^{\times}\big).$$

If τ is a unitary involution on B which is the identity on L, and $u \in L^{\times}$, then $\tau_u := \mathrm{Int}(u) \circ \tau$ is also a unitary involution on B fixing L. Moreover, τ_u is conjugate to τ_v if $uv^{-1} \in F^{\times}\,\mathrm{N}_{\mathrm{KL/L}}\big((KL)^{\times}\big)$. Thus, after a choice of a particular involution τ, we have a morphism of pointed sets from $L^{\times}/F^{\times}\,\mathrm{N}_{\mathrm{KL/L}}\big((KL)^{\times}\big)$ to the set of conjugacy classes of unitary involutions of B which are the identity on L, sending u to τ_u. By Corollary 19.3 of [3], this map is a surjection. So we have a surjective map from $L^{\times}/F^{\times}\,\mathrm{N}_{\mathrm{KL/L}}\big((KL)^{\times}\big)$ to the set of isomorphism classes of triples (B, τ, L).

Theorem 3.5 should say that the fibers of this surjection should be the orbit of the group $\mathrm{Aut}_F(L)$ in $L^{\times}/F^{\times}\,\mathrm{N}_{\mathrm{KL/L}}\big((KL)^{\times}\big)$, corresponding to the $\mathrm{Aut}_F(L)$-orbit of (B, Q). However, to make this statement correct, we need to choose a particular involution τ on B. It is not clear in general what this involution should be. The involution τ should be chosen so that the surface $S(B, \tau, L)$ should correspond to the pair $(B, M_2(L))$. In particular, if $B = M_3(K)$ is split, the surface described by the triple $(M_3(K), \tau, L)$ should have a rational point, by Corollary 3.6. The next remark constructs the involution in this case.

Remark 3.9. Given K and L, we will find a triple (B, τ, L), (i.e. a central simple algebra of degree 3 over K with an involution τ such that $L \subset \mathrm{Sym}(B, \tau)$,) so that the corresponding del Pezzo surface $S(B, \tau, L)$ constructed in [1] has a rational point.

As KL is a three dimensional vector space over K, $B := \mathrm{End}_K(KL)$ is a Azumaya K-algebra of rank 9. Left multiplication by an element of KL determines an embedding of KL into B. If σ is the nontrivial L-automorphism of KL, $h(x, y) = \mathrm{Tr}_{KL/K}(\sigma(x)y)$ defines a hermitian form on KL. This hermitian form on KL induces an involution of the second kind τ on B, such that $L \subset \mathrm{Sym}(B, \tau)$. So we have a triple (B, τ, L). Let S denote the corresponding del Pezzo surface of 6, constructed in [1].

I claim that S contains a rational point. According to [1], it suffices to show that there is a right ideal I of B of reduced dimension 1 such that $(I \cdot \tau(I)) \cap \mathrm{Sym}(B, \tau) \subset F \oplus L^{\perp}$, where $L^{\perp} = \{x \in \mathrm{Sym}(B, \tau)|\, \mathrm{Trd}(lx) = 0,\ \text{for all } l \in L\}$.

Let $W = \operatorname{span}_K(1) \subset LK$, so that W is a one dimensional K-subspace of KL. and then $I = \operatorname{Hom}_K(KL, W)$ is a right ideal of B of reduced dimension 1, generated by the linear map $t = \operatorname{Tr}_{KL/K} : KL \to K \hookrightarrow KL$. We want to show that $t \in \operatorname{Sym}(B, \tau)$. First, note that for any $x \in KL$, $\operatorname{Tr}_{KL/K}(\sigma(x)) = \sigma(\operatorname{Tr}_{LK/L}(x))$. If $x, y \in KL$,

$$
\begin{aligned}
h(x, t(y)) &= \operatorname{Tr}_{KL/K}(\sigma(x)\operatorname{Tr}_{KL/K}(y)) \\
&= \operatorname{Tr}_{KL/K}(\sigma(x))\operatorname{Tr}_{KL/K}(y) \\
&= \operatorname{Tr}_{KL/K}(\sigma(\operatorname{Tr}_{KL/K}(x))y) \\
&= h(t(x), y).
\end{aligned}
$$

So $t \in \operatorname{Sym}(B, \tau)$, which implies that $\tau(I)$ is a left ideal of B, also generated by t. Thus, $I \cdot \tau(I) = tBt$.

In order to prove $I \cdot \tau(I) \cap \operatorname{Sym}(B, \tau) \subset F \oplus L^{\perp}$, it suffices to consider the case where $L = F^3$ is split. So we can choose a basis e_1, e_2, e_3 of idempotents for KL over K. In this basis, τ is the standard adjoint involution, $\operatorname{Sym}(B, \tau)$ is the set of hermitian matrices, and $F \oplus L^{\perp}$ is the set of hermitian matrices where the diagonal entries agree. Moreover, t is the matrix with ones in every entry, so $t \in F \oplus L^{\perp}$. A direct calculation shows $I \cdot \tau(I) = \operatorname{span}_K(t)$, and hence $(I \cdot \tau(I)) \cap \operatorname{Sym}(B, \tau) = \operatorname{span}_F(t) \subset F \oplus L^{\perp}$.

4. K_0 OF DEL PEZZO SURFACES

Let $S = S(B, Q, KL)$ be a del Pezzo surface of degree 6, a T-toric variety for a two dimensional torus T. Let $Z \subset S$ be the closed variety such that $\overline{Z}$ is the union of six lines l_0, l_1, l_2, m_0, m_1, and m_2. Recall the exact sequence (2) from Section 2, where $\mathbb{Z}[KL/F]$ is the lattice of connected components of $\overline{Z} \subset \overline{S}$, the lines l_i and m_i, and the homomorphism $\mathbb{Z}[KL/F] \to \operatorname{Pic}(\overline{S})$ sends each line to the corresponding invertible sheaf on $\overline{S}$. From the exact sequence (1), we see that $\widehat{T}$ is the subgroup of $\mathbb{Z}[KL/F]$ generated by $l_0 - l_1 - (m_0 - m_1)$, $l_0 - l_2 - (m_0 - m_2)$, and $l_1 - l_2 - (m_1 - m_2)$. Note that any one of these 3 generators can be expressed as a linear combination of the other 2. So $\operatorname{Pic}(\overline{S})$ is generated by the invertible sheaves $\mathcal{L}(-l_i)$, $\mathcal{L}(-m_j)$, and we have that the invertible sheaves $\mathcal{L}(-l_i - m_j)$ and $\mathcal{L}(-l_j - m_i)$ are isomorphic for $i, j = 0, 1, 2$.

There is another way to recover these generators and relations, which does not depend on the theory of toric varieties. Recall that there is a morphism $p_1 : \overline{S} \to \mathbb{P}^2$, obtained by blowing down the lines m_0, m_1, and m_2. If x_0, x_1, x_2 are the homogeneous coordinates of $\mathbb{P}^2$ and $D_i = \{x_i = 0\}$ for $i = 0, 1, 2$, then D_0, D_1, and D_2 are all linearly equivalent divisors on $\mathbb{P}^2$, and thus their strict transforms $m_1 + l_0 + m_2$, $m_0 + l_1 + m_2$, and $m_0 + l_2 + m_1$ are all linearly equivalent divisors on $\overline{S}$. Therefore the corresponding invertible sheaves $\mathcal{L}(-m_1 - l_0 - m_2)$, $\mathcal{L}(-m_0 - l_1 - m_2)$, and $\mathcal{L}(-m_0 - l_2 - m_1)$ on $\overline{S}$ are isomorphic. From this we can conclude that $\operatorname{Pic}(\overline{S})$ is generated by the invertible sheaves $\mathcal{L}(-m_0)$, $\mathcal{L}(-m_1)$, $\mathcal{L}(-m_2)$, $\mathcal{L}(-m_1 - l_0 - m_2)$, $\mathcal{L}(-m_0 - l_1 - m_2)$, and $\mathcal{L}(-m_0 - l_2 - m_1)$, and we have that $\mathcal{L}(-l_i - m_j - l_k)$ and $\mathcal{L}(-l_j - m_i - l_k)$ are isomorphic for any

$i, j = 0, 1, 2$. This presentation is equivalent to that in the previous paragraph. Similarly, this presentation can be obtained by considering the morphism $p_2 : \overline{S} \to \mathbb{P}^2$ obtained by blowing down the lines l_0 l_1, and l_2.

We define the following locally free sheaves on $\overline{S}$:

$$\mathcal{I}_1 = \mathcal{L}(-m_1 - l_0 - m_2) \oplus \mathcal{L}(-m_0 - l_1 - m_2) \oplus \mathcal{L}(-m_0 - l_2 - m_1)$$
$$\mathcal{I}_2 = \mathcal{L}(-l_1 - m_0 - l_2) \oplus \mathcal{L}(-l_0 - m_1 - l_2) \oplus \mathcal{L}(-l_0 - m_2 - l_1)$$
$$\mathcal{J}_1 = \mathcal{L}(-l_0 - m_1) \oplus \mathcal{L}(-l_1 - m_0)$$
$$\mathcal{J}_2 = \mathcal{L}(-l_0 - m_2) \oplus \mathcal{L}(-l_2 - m_0)$$
$$\mathcal{J}_3 = \mathcal{L}(-l_1 - m_2) \oplus \mathcal{L}(-l_2 - m_1).$$

The Γ-action on the hexagon of lines induces an action on the locally free sheaves $\mathcal{I}_1 \oplus \mathcal{I}_2$ and $\mathcal{J}_1 \oplus \mathcal{J}_2 \oplus \mathcal{J}_3$, compatible with the action on $\overline{S}$. Therefore $\mathcal{I}_1 \oplus \mathcal{I}_2$ and $\mathcal{J}_1 \oplus \mathcal{J}_2 \oplus \mathcal{J}_3$ descend to sheaves $\mathcal{I}$ and $\mathcal{J}$ on S.

We will consider the following endomorphism rings: $B' = \operatorname{End}_{\mathcal{O}_S}(\mathcal{I})^{\mathrm{op}}$, and $Q' = \operatorname{End}_{\mathcal{O}_S}(\mathcal{I})^{\mathrm{op}}$. As S is projective, $\operatorname{End}_{\mathcal{O}_S}(\mathcal{O}_S)^{\mathrm{op}} = F$, and since $\mathcal{I}$ and $\mathcal{J}$ are $\mathcal{O}_S$-modules, it follows that B' and Q' are F-algebras. For i, j, and k not equal, $\operatorname{End}_{\mathcal{O}_{\overline{S}}}(\mathcal{L}(-m_i - l_j - m_k)) = \operatorname{End}_{\mathcal{O}_{\overline{S}}}(\mathcal{L}(-l_i - m_j - l_k)) = \overline{F}$, so we see that $\overline{F}^6$ embeds diagonally into $\operatorname{End}_{\mathcal{O}_{\overline{S}}}(\mathcal{I}_1 \oplus \mathcal{I}_2)$. Moreover, since $\mathcal{L}(-l_i - m_j)$ and $\mathcal{L}(-l_j - m_i)$ are isomorphic for any i, j, $\mathcal{I}_1 \oplus \mathcal{I}_2 = (\mathcal{L}(-m_1 - l_0 - m_2) \oplus \mathcal{L}(-l_1 - m_0 - l_2)) \otimes_{\overline{F}} V$, where V is an $\overline{F}$-vector space of dimension 3. An element of $\operatorname{Hom}_{\mathcal{O}_{\overline{S}}}(\mathcal{L}(-m_1 - l_0 - m_2), \mathcal{L}(-l_1 - m_0 - l_2))$ is given by a global section of $\mathcal{L}(m_2 - l_2)$. Any non-zero global section of $\mathcal{L}(m_2 - l_2)$ would give a function defined on a neighborhood of $l_2 \subset \overline{S}$ with vanishing set l_2. Blowing down the lines l_i, this function would then correspond to a function defined on an open subset of $\mathbb{P}^2$ with vanishing set a point, which is impossible, since a point is a codimension 2 subvariety of $\mathbb{P}^2$. Thus $\mathcal{L}(m_2 - l_2)$ has no nonzero global sections. Similarly $\operatorname{Hom}_{\mathcal{O}_{\overline{S}}}(\mathcal{L}(-l_1 - m_0 - l_2), \mathcal{L}(-m_1 - l_0 - m_2)) = 0$, and so $\operatorname{End}_{\mathcal{O}_{\overline{S}}}(\mathcal{I}_1 \oplus \mathcal{I}_2) = \operatorname{End}_{\mathcal{O}_{\overline{S}}}(\mathcal{L}(-m_i - l_j - m_k) \times \mathcal{L}(-l_i - m_j - l_k)) \otimes_{\overline{F}} \operatorname{End}_{\overline{F}}(V) = \operatorname{End}_{\overline{F}^2}(V_{\overline{F}^2})$. So the center of $\operatorname{End}_{\mathcal{O}_{\overline{S}}}(\mathcal{I}_1 \oplus \mathcal{I}_2)$ is a copy of $\overline{F}^2$, contained in $\overline{F}^6$. This chain $\overline{F}^2 \subset \overline{F}^6 \subset \operatorname{End}_{\mathcal{O}_{\overline{S}}}(\mathcal{I}_1 \oplus \mathcal{I}_2)$ descends to $K \subset KL \subset \operatorname{End}_{\mathcal{O}_S}(\mathcal{I})$, with K the center of $\operatorname{End}_{\mathcal{O}_S}(\mathcal{I})$.

Now let E be any separable field extension of F over which the lines l_i, m_j are defined. This is equivalent to E splitting both K and L. the above arguments show that $\operatorname{End}_{\mathcal{O}_S}(\mathcal{I}) \otimes_F E \approx M_3(E^2)$, where $E^2 \approx K \otimes_F E$. Therefore, we conclude that B' is an Azumaya K-algebra of rank 9 which contains KL as a subalgebra. A similar argument shows that Q' is an Azumaya L-algebra of rank 4 which also contains a copy of KL.

Theorem 4.1. $B' = \operatorname{End}_{\mathcal{O}_S}(\mathcal{I})^{\mathrm{op}}$ and B are isomorphic as K-algebras. Similarly, $Q' = \operatorname{End}_{\mathcal{O}_S}(\mathcal{J})^{\mathrm{op}}$ and Q are isomorphic as L-algebras.

Proof. Let $\widetilde{S}$ be as in Proposition 2.1, and let $\widetilde{\mathcal{I}}$ and $\widetilde{\mathcal{J}}$ be the sheaves associated to $\widetilde{S}$ as above. Twisting $\widetilde{\mathcal{I}}$ and $\widetilde{\mathcal{J}}$ by the Γ-action corresponding to the pair

(K, L), we get sheaves $\mathcal{I}_0$ and $\mathcal{J}_0$ associated to the T-toric model S_0. Let $B'_0 = \mathrm{End}_{\mathcal{O}_S}(\mathcal{I}_0)^{\mathrm{op}}$ and $Q'_0 = \mathrm{End}_{\mathcal{O}_S}(\mathcal{J}_0)^{\mathrm{op}}$. The embeddings of KL into B'_0 and Q'_0 described above induce the following commutative diagrams (cf. (3)):

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & R^{(1)}_{K/F}(\mathbb{G}_m) & \longrightarrow & \mathbf{G}_L & \longrightarrow & T & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & R_{K/F}(\mathbb{G}_m) & \longrightarrow & R_{K/F}(\mathbf{GL}(B'_0)) & \longrightarrow & R_{K/F}(\mathbf{PGL}(B'_0)) & \longrightarrow & 1,
\end{array}
$$

and

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & R^{(1)}_{L/F}(\mathbb{G}_m) & \longrightarrow & \mathbf{G}_K & \longrightarrow & T & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & R_{L/F}(\mathbb{G}_m) & \longrightarrow & R_{L/F}(\mathbf{GL}(Q'_0)) & \longrightarrow & R_{L/F}(\mathbf{PGL}(Q'_0)) & \longrightarrow & 1.
\end{array}
$$

The first diagram induces the following commutative diagram of cohomology sets:

$$
\begin{array}{ccc}
H^1(F, T) & \longrightarrow & \mathrm{Ker}(\mathrm{cor}_{K/F} : \mathrm{Br}(K) \to \mathrm{Br}(F)) \\
\downarrow & & \downarrow \\
1 \longrightarrow H^1(K, \mathbf{PGL}(B'_0)) & \longrightarrow & \mathrm{Br}(K).
\end{array}
$$

The left vertical arrow sends the triple (B, Q, KL) to the endomorphism ring $B' = \mathrm{End}(\mathcal{I})^{\mathrm{op}}$, where $\mathcal{I}$ is the sheaf associated to the surface $S(B, Q, KL)$ constructed above. The upper horizontal arrow sends the triple (B, Q, KL) to the class of $[B] \in \mathrm{Br}(K)$, which lands in the subgroup of elements of trivial norm. The right vertical map is the inclusion homomorphism. The lower horizontal arrow sends a K-algebra to its corresponding element in $\mathrm{Br}(K)$. By the commutativity of the diagram, we see that $[B] = [B']$ in $\mathrm{Br}(K)$. But these algebras have the same rank, so they must be isomorphic as K-algebras. A similar argument shows that $Q' = \mathrm{End}_{\mathcal{O}_S}(\mathcal{J})^{\mathrm{op}}$ and Q are isomorphic as L-algebras. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let A be the separable F-algebra $F \times B \times Q$, so that $\mathbf{P}(A) = \mathbf{P}(F) \times \mathbf{P}(B) \times \mathbf{P}(Q)$. Define the exact functors u_F from $\mathbf{P}(F)$ to $\mathbf{P}(S)$ by $M_1 \mapsto \mathcal{O}_S \otimes_F M_1$, u_B from $\mathbf{P}(B)$ to $\mathbf{P}(S)$ by $M_2 \mapsto \mathcal{I} \otimes_B M_2$, and u_Q from $\mathbf{P}(Q)$ to $\mathbf{P}(S)$ by $M_3 \mapsto \mathcal{J} \otimes_Q M_3$. If we set $\mathcal{P} = \mathcal{O}_S \oplus \mathcal{I} \oplus \mathcal{J}$, then the respective right actions of F, B, and Q on $\mathcal{O}_S$, $\mathcal{I}$, and $\mathcal{J}$ combine to give a right action of $A = F \times B \times Q$ on $\mathcal{P}$. Therefore, we can define an exact functor from $\mathbf{P}(A)$ to $\mathbf{P}(S)$ by sending M to $\mathcal{P} \otimes_A M$. This exact functor induces a homomorphism:

$$
\phi : K_0(A) \to K_0(S).
$$

More generally, if Y is any F-variety, then we have an exact functor from $\mathbf{P}(Y; A)$ to $\mathbf{P}(Y \times S)$, sending M to $p_2^*(\mathcal{P}) \otimes_{\mathcal{O}_{Y \times S} \otimes_F A} p_1^*(M)$, where $p_1 : Y \times S \to Y$ and $p_2 : Y \times S \to S$ are the projection morphisms. This induces a

homomorphism $\phi_Y : K_0(Y; A) \to K_0(Y \times S)$. Furthermore, if E is any field extension of F, then ϕ naturally extends to a homomorphism $\phi_E : K_0(A_E) \to K_0(S_E)$.

Theorem 4.2. $\phi : K_0(A) \to K_0(S)$ *is an isomorphism.*

We will prove this in several stages. Let us first consider the case where F is separably closed. By Proposition 2.1, S is isomorphic to the blow up of the projective plane at the 3 non-collinear points $[1 : 0 : 0]$, $[0 : 1 : 0]$, and $[0 : 0 : 1]$. Recall that we have the filtration $0 = K_0(S)^{(3)} \subset K_0(S)^{(2)} \subset K_0(S)^{(1)} \subset K_0(S)^{(0)} = K_0(S)$ by codimension of support, and homomorphisms $CH^i(S) \to K_0(S)^{(i/i+1)}$, which send the class of a subvariety V to the equivalence class $[\mathcal{O}_V]$. These homomorphisms are isomorphisms for $i = 0, 1, 2$. So by Proposition 2.1, if P is a rational point of S, $K_0(S)$ is generated by $[\mathcal{O}_S]$, $[\mathcal{O}_{l_0}]$, $[\mathcal{O}_{l_1}]$, $[\mathcal{O}_{l_2}]$, $[\mathcal{O}_{m_0}]$, $[\mathcal{O}_{m_1}]$, $[\mathcal{O}_{m_2}]$, and $[\mathcal{O}_P]$. Moreover, as $CH^0(S)$, $CH^1(S)$, and $CH^2(S)$ are free abelian groups with ranks 1, 4, and 1, respectively, $K_0(S)$ is free abelian with rank 6. Since F is separably closed, K, L, B, and Q are split, and thus $K_0(A)$ is also free abelian of rank 6. Therefore ϕ will be an isomorphism provided it is surjective. So it suffices to show that $[\mathcal{O}_S]$, $[\mathcal{O}_{l_0}]$, $[\mathcal{O}_{l_1}]$, $[\mathcal{O}_{l_2}]$, $[\mathcal{O}_{m_0}]$, $[\mathcal{O}_{m_1}]$, $[\mathcal{O}_{m_2}]$, and $[\mathcal{O}_P]$ are in the image of ϕ.

Clearly, $[\mathcal{O}_S] = [\mathcal{O}_S \otimes_F F]$ is in the image of ϕ. As F is separably closed, $\mathcal{I} = (\mathcal{L}(-m_1 - l_0 - m_2) \oplus \mathcal{L}(-l_1 - m_0 - l_2)) \otimes_F V$, where V is an F-vector space of dimension 3, $K = F \times F \cong \mathrm{End}_{\mathcal{O}_S}(\mathcal{L}(-m_1 - l_0 - m_2)) \times \mathrm{End}_{\mathcal{O}_S}(\mathcal{L}(-l_1 - m_0 - l_2))$, and $\mathrm{End}_{\mathcal{O}_S}(\mathcal{I})^{\mathrm{op}} \cong \mathrm{End}_{F^2}(V_{F^2})$.

Now $\mathrm{Hom}_{F^2}(V_{F^2}, F \times 0)$ is a right $\mathrm{End}_{\mathcal{O}_S}(\mathcal{I})$-module, and thus a left A-module, where the F and Q component of $A = F \times B \times Q$ act trivially. Therefore,

$$\phi\Big(\mathrm{Hom}_{F^2}(V_{F^2}, F \times 0) \Big) = \Big[\mathcal{I} \otimes_B \mathrm{Hom}_{F^2}(V_{F^2}, F \times 0) \Big]$$

$$= \Big[\Big(\mathcal{L}(-m_1 - l_0 - m_2) \oplus \mathcal{L}(-l_1 - m_0 - l_2) \Big)$$

$$\otimes_{F^2} (F \times 0) \Big]$$

$$= [\mathcal{L}(-m_1 - l_0 - m_2)],$$

where we use Morita equivalence in the second line. A mirror argument shows that $\phi\Big(\mathrm{Hom}_{F^2}(V_{F^2}, 0 \times F) \Big) = [\mathcal{L}(-l_1 - m_0 - l_2)]$, and a similar argument applied to $\mathcal{J}$ and Q shows that $[\mathcal{L}(-l_0 - m_1)]$, $[\mathcal{L}(-l_0 - m_2)]$, and $[\mathcal{L}(-l_1 - m_2)]$ are in the image of ϕ.

Now let $i, j \in \{0, 1, 2\}$ and not equal. By Proposition 2.1, the lines l_i and m_j have intersection a rational point P of S, with multiplicity 1, the lines m_i and m_j are skew, and the lines l_i and l_j are skew. Thus we have the following resolutions of $\mathcal{O}_P$ and $\mathcal{O}_S$:

$$0 \to \mathcal{L}(-l_i - m_j) \xrightarrow{\begin{pmatrix} \otimes \mathcal{L}(m_j) \\ \otimes \mathcal{L}(l_i) \end{pmatrix}} \mathcal{L}(-l_i) \oplus \mathcal{L}(-m_j) \xrightarrow{(\otimes \mathcal{L}(l_i), -\otimes \mathcal{L}(m_j))} \mathcal{O}_S \to \mathcal{O}_P \to 0,$$

$$0 \to \mathcal{L}(-m_i - m_j) \xrightarrow{\begin{pmatrix} \otimes \mathcal{L}(m_i) \\ \otimes \mathcal{L}(m_j) \end{pmatrix}} \mathcal{L}(-m_i) \oplus \mathcal{L}(-m_j) \xrightarrow{(\otimes \mathcal{L}(m_i), -\otimes \mathcal{L}(m_j))} \mathcal{O}_S \to 0,$$

and

$$0 \to \mathcal{L}(-l_i - l_j) \xrightarrow{\begin{pmatrix} \otimes \mathcal{L}(l_i) \\ \otimes \mathcal{L}(l_j) \end{pmatrix}} \mathcal{L}(-l_i) \oplus \mathcal{L}(-l_j) \xrightarrow{(\otimes \mathcal{L}(l_i), -\otimes \mathcal{L}(l_j))} \mathcal{O}_S \to 0.$$

In addition, when $D = l_i$ or m_i, we have the standard resolution

$$0 \to \mathcal{L}(-D) \xrightarrow{\otimes \mathcal{L}(D)} \mathcal{O}_S \to \mathcal{O}_D \to 0.$$

So $0 = [\mathcal{O}_S] - [\mathcal{L}(-m_i)] - [\mathcal{L}(-m_j)] + [\mathcal{L}(-m_i - m_j)]$ in $K_0(S)$. If we take $k \in \{0, 1, 2\}$ not equal to i or j and multiply this equation by $[\mathcal{L}(-l_k)]$, we see that

$$0 = [\mathcal{L}(-l_k)] - [\mathcal{L}(-l_k - m_i)] - [\mathcal{L}(-l_k - m_j)] + [\mathcal{L}(-m_i - l_k - m_j)].$$

Therefore,

$$\begin{aligned} [\mathcal{O}_{l_k}] &= [\mathcal{O}_S] - [\mathcal{L}(-l_k)] \\ &= [\mathcal{O}_S] - [\mathcal{L}(-l_k - m_i)] - [\mathcal{L}(-l_k - m_j)] + [\mathcal{L}(-m_i - l_j - m_k)] \end{aligned}$$

is in the image of ϕ. The same argument with l and m interchanged shows that $[\mathcal{O}_{m_k}]$ is in the image of ϕ for $k \in \{0, 1, 2\}$. Finally, $[\mathcal{O}_P] = [\mathcal{O}_S] - [\mathcal{L}(-l_0)] - [\mathcal{L}(-m_1)] + [\mathcal{L}(-l_0 - m_1)]$ is in the image of ϕ, and thus ϕ is surjective when F is separably closed.

Proposition 4.3. $\phi : K_0(A) \to K_0(S)$ *is an isomorphism if B and Q are split.*

Proof. By the preceding argument, $\phi_{\overline{F}} : K_0(\overline{A}) \to K_0(\overline{S})$ is an isomorphism. Moreover, $\phi_{\overline{F}}$ commutes with the action of Γ on both $K_0(\overline{A})$ and $K_0(\overline{S})$, and thus it descends to an isomorphism on the Γ-invariant subgroups. Therefore, we have the following commutative diagram:

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\phi} & K_0(S) \\ \downarrow & & \downarrow \\ K_0(\overline{A})^\Gamma & \xrightarrow{\phi_{\overline{F}}} & K_0(\overline{S})^\Gamma. \end{array}$$

As $\phi_{\overline{F}}$ and the left vertical map $K_0(A) \to K_0(\overline{A})^\Gamma$ are isomorphisms, ϕ must be injective. Moreover, if the right vertical map is injective, then ϕ is surjective,

and hence an isomorphism. So it suffices to show that $K_0(S) \to K_0(\overline{S})^\Gamma$ is injective.

To see this, note that the rank and wedge homomorphisms $\mathrm{rank} : K_0(\overline{S}) \to \mathbb{Z}$ and $\wedge : K_0(\overline{S})^{(1)} \to \mathrm{Pic}(\overline{S})$ commute with the action of Γ. Thus we have the following short exact sequences of Γ-modules:

$$0 \to K_0(\overline{S})^{(2)} \to K_0(\overline{S})^{(1)} \xrightarrow{\wedge} \mathrm{Pic}(\overline{S}) \to 0$$

and

$$0 \to K_0(\overline{S})^{(1)} \to K_0(\overline{S}) \xrightarrow{\mathrm{rank}} \mathbb{Z} \to 0.$$

These sequences of Γ-modules induce the following long exact sequences:

$$0 \to (K_0(\overline{S})^{(2)})^\Gamma \to (K_0(\overline{S})^{(1)})^\Gamma \xrightarrow{\wedge} \mathrm{Pic}(\overline{S})^\Gamma \to H^1(F, K_0(\overline{S})^{(2)})$$

and

$$0 \to (K_0(\overline{S})^{(1)})^\Gamma \to (K_0(\overline{S}))^\Gamma \xrightarrow{\mathrm{rank}} \mathbb{Z} \to H^1(F, K_0(\overline{S})^{(1)}).$$

The map $K_0(S) \to K_0(\overline{S})^\Gamma$ induces the following commutative diagrams:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K_0(S)^{(2)} & \longrightarrow & K_0(S)^{(1)} & \xrightarrow{\wedge} & \mathrm{Pic}(S) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (K_0(\overline{S})^{(2)})^\Gamma & \longrightarrow & (K_0(\overline{S})^{(1)})^\Gamma & \xrightarrow{\wedge} & \mathrm{Pic}(\overline{S})^\Gamma & \longrightarrow & H^1(F, K_0(\overline{S})^{(2)}),
\end{array}
$$

and

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K_0(S)^{(1)} & \longrightarrow & K_0(S) & \xrightarrow{\mathrm{rank}} & \mathbb{Z} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow = & & \\
0 & \longrightarrow & (K_0(\overline{S})^{(1)})^\Gamma & \longrightarrow & K_0(\overline{S})^\Gamma & \xrightarrow{\mathrm{rank}} & \mathbb{Z} & \longrightarrow & H^1(F, K_0(\overline{S})^{(1)}).
\end{array}
$$

As B and Q are split, S has a rational point by Corollary 3.6. Thus the homomorphism $K_0(S)^2 \to (K_0(\overline{S})^{(2)})^\Gamma$ is a surjective homomorphism of free abelian groups of rank 1, and therefore an isomorphism. Moreover, the homomorphism $\mathrm{Pic}(S) \to \mathrm{Pic}(\overline{S})^\Gamma$ is injective. Thus, by applying the Snake Lemma to the first diagram and then to the second, we see that $K_0(S) \to (K_0(\overline{S}))^\Gamma$ is injective. $\square$

Remark 4.4. If P and P' are rational points of $\overline{S}$, they define equal classes in $\mathrm{CH}^2(\overline{S})$ by Proposition 2.1, and hence $[\mathcal{O}_P] = [\mathcal{O}_{P'}]$ in $K_0(\overline{S})^{(2)}$. So $K_0(\overline{S})^{(2)}$ is generated by the Γ-invariant element $[\mathcal{O}_P]$, and thus is a trivial Γ-module. Similarly, our arguments above show that the set $\{[\mathcal{O}_{l_0} \oplus \mathcal{O}_{m_1}], [\mathcal{O}_{l_0} \oplus \mathcal{O}_{m_2}], [\mathcal{O}_{l_1} \oplus \mathcal{O}_{m_2}], [\mathcal{O}_{m_1} \oplus \mathcal{O}_{l_0} \oplus \mathcal{O}_{m_2}], [\mathcal{O}_{l_1} \oplus \mathcal{O}_{m_0} \oplus \mathcal{O}_{l_2}]\}$ is a Γ-invariant basis of $K_0(\overline{S})^{(1)}$, and thus $K_0(\overline{S})^{(1)}$ is a permutation module. It follows that $H^1(F, K_0(\overline{S})^{(2)}) = H^1(F, K_0(\overline{S})^{(1)}) = 0$.

We will need the following proposition (cf. Proposition 6.1 of [6]).

Proposition 4.5. *If Y is a variety such that the homomorphism $\phi_{F(y)}$: $K_0(A_{F(y)}) \to K_0(S_{F(y)})$ is an isomorphism for every $y \in Y$, then $\phi_Y : K_0(Y; A) \to K_0(S \times Y)$ is surjective.*

Proof. We do this by double induction on the dimension of Y and the number of irreducible components of Y.

If Y has a proper irreducible component Y', with complement U, we have the following localization exact sequence (cf. [8]):

$$
\begin{array}{ccccccc}
K_0(Y'; A) & \longrightarrow & K_0(Y; A) & \longrightarrow & K_0(U; A) & \longrightarrow & 0 \\
\downarrow{\phi_{Y'}} & & \downarrow{\phi_Y} & & \downarrow{\phi_U} & & \\
K_0(Y' \times S) & \longrightarrow & K_0(Y \times S) & \longrightarrow & K_0(U \times S) & \longrightarrow & 0
\end{array}
$$

By our inductive assumption, the vertical maps on the right and on the left are surjective. This implies that the middle vertical map is surjective as well. So we may assume that Y is irreducible. If Y is not reduced, then we the natural $Y_{\mathrm{red}} \to Y$ induces the commutative diagram

$$
\begin{array}{ccc}
K_0(Y_{\mathrm{red}}; A) & \longrightarrow & K_0(Y; A) \\
\downarrow{\phi_{Y_{\mathrm{red}}}} & & \downarrow{\phi_Y} \\
K_0(Y_{\mathrm{red}} \times S) & \longrightarrow & K_0(Y \times S),
\end{array}
$$

where the horizontal arrows are isomorphisms. Thus we may also assume that Y is reduced.

Now let $x \in K_0(Y \times S)$. By assumption, $\phi_{F(Y)} : K_0(A_{F(Y)}) \to K_0(S_{F(Y)})$ is an isomorphism, and hence there exists an open set U in Y such the image of x in $K_0(U \times S)$ is in the image of ϕ_U. We again consider the localization exact sequence:

$$
\begin{array}{ccccccc}
K_0(Z; A) & \longrightarrow & K_0(Y; A) & \longrightarrow & K_0(U; A) & \longrightarrow & 0 \\
\downarrow{\phi_Z} & & \downarrow{\phi_Y} & & \downarrow{\phi_U} & & \\
K_0(Z \times S) & \longrightarrow & K_0(Y \times S) & \longrightarrow & K_0(U \times S) & \longrightarrow & 0,
\end{array}
$$

where Z is the complement of U in Y. By assumption, Z has a strictly smaller dimension than Y, and so by our inductive hypothesis, ϕ_Z is surjective. A standard diagram chase shows that $x \in \mathrm{Im}(\phi_Y)$. $\qquad\square$

Proof of Theorem 4.2. Let $SB(B)$ be the Severi-Brauer K-variety associated to B, $SB(Q)$ be the Severi-Brauer L-variety associated to Q, and $Y = R_{K/F}(SB(B)) \times R_{L/F}(SB(Q))$ be the product of the restriction of scalars of both varieties. Then for any field extension E of F, $Y(E)$ is nonempty if and only if $SB(B)(K \otimes_F E)$ and $SB(Q)(L \otimes_F E)$ are nonempty if and only if $B_E = B \otimes_K (K \otimes_F E)$ and $Q_E = Q \otimes_L (L \otimes_F E)$ are split.

The projection $p : Y \to \mathrm{Spec}(F)$ induce the following diagram:

$$
\begin{array}{ccc}
K_0(A) & \xrightarrow{\phi} & K_0(S) \\
{\scriptstyle p_*}\Big\uparrow\Big\downarrow{\scriptstyle p^*} & & {\scriptstyle (p_Y)_*}\Big\uparrow\Big\downarrow{\scriptstyle (p_Y)^*} \\
K_0(Y;A) & \xrightarrow{\phi_Y} & K_0(Y \times S),
\end{array}
$$

where $p_Y : Y \times S \to S$ is the projection induced by p. Both squares commute, and $p_* p^*$ is the identity homomorphism, as Y is a geometrically rational variety. For every $y \in Y$, $Y(F(y)) \neq \emptyset$, so $B_{F(y)}$ and $Q_{F(y)}$ are split, and thus $\phi_{F(y)} : K_0(A_{F(y)}) \to K_0(S_{F(y)})$ is an isomorphism by Proposition 4.3. So by Proposition 4.5, $\phi_Y : K_0(Y;A) \to K_0(Y \times S)$ is surjective. A diagram chase shows the top horizontal map ϕ is also surjective.

Now let E be any field extension of F such that $S(E) \neq \emptyset$. Then, B_E and Q_E are split, and so $\phi_E : K_0(A_E) \to K_0(S_E)$ is an isomorphism, again by Proposition 4.3. The homomorphisms ϕ and ϕ_E fit into the following commutative diagram:

$$
\begin{array}{ccc}
K_0(A) & \xrightarrow{\phi} & K_0(S) \\
\Big\downarrow & & \Big\downarrow \\
K_0(A_E) & \xrightarrow{\phi_E} & K_0(S_E),
\end{array}
$$

where the vertical homomorphisms are induced by the inclusion $F \subset E$. The bottom horizontal map is an isomorphism, and the left vertical map is injective. It follows that ϕ is injective, and hence an isomorphism. $\square$

As ϕ is an isomorphism, the hypothesis on the variety V in Proposition 4.5 is always true, we obtain the following corollary.

Corollary 4.6. $\phi_V : K_0(V;A) \to K_0(V \times S)$ *is surjective for any F-variety* V. $\square$

5. Higher K-theory

We have shown that the K_0 groups of S and A coincide. We will show that this is also true for the higher Quillen K-groups.

For any F-varieties X, Y, and Z, and separable F-algebras A, B, and C, consider the functor:

$$\mathbf{P}(Y \times Z; B^{\mathrm{op}} \otimes_F C) \times \mathbf{P}(X \times Y; A^{\mathrm{op}} \otimes_F B) \to \mathbf{P}(X \times Z; A^{\mathrm{op}} \otimes_F C),$$

sending a pair (M, N) to $(p_{13})_*(p_{23}^*(M) \otimes_B p_{12}^*(N))$, where p_{12}, p_{23}, and p_{13} are the projections of $X \times Y \times Z$ onto its factors. This functor is bi-exact, and thus induces a product map

$$K_n(Y \times Z; B^{\mathrm{op}} \otimes_F C) \otimes_{\mathbb{Z}} K_m(X \times Y; A^{\mathrm{op}} \otimes_F B) \to K_{n+m}(X \times Z; A^{\mathrm{op}} \otimes_F C).$$

We will denote the image of $u \otimes x$ under this map by $u \bullet_B x$.

We recall the K-Motivic Category $\mathcal{C}$ and some of its properties. The details can be found in [6] and [7]. Objects of $\mathcal{C}$ are pairs (X, A), where X is an

F-variety and A is a separable F-algebra. For two pairs (X, A) and (Y, B) in $\mathcal{C}$, we set $\mathrm{Mor}_{\mathcal{C}}((X, A), (Y, B)) := K_0(X \times Y; A^{\mathrm{op}} \otimes_F B)$. The composition law is $g \circ f = g \bullet_B f$, for $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (Z, C)$ in $\mathcal{C}$. For any pair (X, A) with X smooth, the identity element $1_{(X,A)} \in K_0(X \times X; A^{\mathrm{op}} \otimes_F A)$ is the element $[\mathcal{O}_\Delta \otimes_F A]$, where $\Delta \subset X \times X$ is the diagonal. For any F-variety X and any separable F-algebra A, we will write X for the pair (X, F) and A for the pair $(\mathrm{Spec}\, F, A)$. Finally, for any F-variety V and any nonnegative integer n, we have a realization functor K_n^V, which sends an object (X, A) to $K_n(V \times X; A)$, and $K_n^V(f)(x) = f \bullet_A x \in K_n(V \times Y; B)$ for any morphism $f \in \mathrm{Mor}_{\mathcal{C}}((X, A), (Y, B)) = K_0(X \times Y; A^{\mathrm{op}} \otimes_F B)$ and $x \in K_n(V \times X; A)$. We will denote $K_n^{\mathrm{Spec}\, F}$ by K_n.

As we mentioned in the beginning of Section 4, there is a left action of $A^{\mathrm{op}} = \mathrm{End}_{\mathcal{O}_S}(\mathcal{O}_S) \times \mathrm{End}_{\mathcal{O}_S}(\mathcal{I}) \times \mathrm{End}_{\mathcal{O}_S}(\mathcal{J})$ on the locally free sheaf $\mathcal{P} = \mathcal{O}_S \oplus \mathcal{I} \oplus \mathcal{J}$. So $\mathcal{P} \in \mathbf{P}(X; A^{\mathrm{op}})$. The corresponding element $[\mathcal{P}] \in K_0(S; A^{\mathrm{op}})$ defines a morphism $u : A \to S$ in $\mathcal{C}$. It follows from the construction of the realization functor that $\phi_V = K_0^V(u)$ for any V. In particular, $K_0(u) = \phi$.

Theorem 5.1. $u : A \to S$ *is an isomorphism in* $\mathcal{C}$.

Proof. Let V be any F-variety. Equating $K_0(V; A)$ (resp. $K_0(V \times S)$) with $\mathrm{Mor}_{\mathcal{C}}(V, A)$ (resp. $\mathrm{Mor}_{\mathcal{C}}(V, S)$), $K_0^V(u) : K_0(V; A) \to K_0(V \times S)$ is just post-composition in $\mathcal{C}$ with u. By Corollary 4.6, $K_0^V(u) = \phi_V$ is surjective for any variety V. In particular, if $V = S$, there is an element $v \in K_0(S; A)$ such that $uv = [\mathcal{O}_\Delta]$, i.e. u has a right inverse v in $\mathcal{C}$.

We want to show that v is also a left inverse to u in $\mathcal{C}$, i.e. $vu = [A] \in K_0(A^{\mathrm{op}} \otimes_F A)$. As $K_0(A^{\mathrm{op}} \otimes_F A) \hookrightarrow K_0((A^{\mathrm{op}} \otimes_F A)_{\overline{F}}) = K_0(A_{\overline{F}}^{\mathrm{op}} \otimes_{\overline{F}} A_{\overline{F}})$, it suffices to consider the case where K, L, B and Q are split. So $A = F \times M_3(F \times F) \times M_2(F \times F \times F)$, and thus A is isomorphic in $\mathcal{C}$ to $F \times (F \times F) \times (F \times F \times F)$ (cf. example 1.6 of [6]). So $K_0(A) \cong \mathbb{Z}^6$, and $\mathrm{Mor}_{\mathcal{C}}(A, A) = K_0(A^{\mathrm{op}} \otimes_F A) \cong M_6(\mathbb{Z})$. Moreover, under this isomorphism $[A] \in K_0(A^{\mathrm{op}} \otimes_F A)$ corresponds to the identity matrix.

So $vu \in K_0(A^{\mathrm{op}} \otimes_F A)$ is represented by a matrix M with integer entries. It follows that the corresponding homomorphism $K_0(vu)$ from $K_0(A) \cong \mathbb{Z}^6$ to itself is multiplication by this matrix M. Now, as v is a right inverse to u in $\mathcal{C}$, $K_0(v)$ is a right inverse to $K_0(u)$. However, $K_0(u) = \phi$ is an isomorphism by Theorem 4.2, so in fact $K_0(v) = K_0(u)^{-1}$. Thus $K_0(vu) = K_0(v)K_0(u) = \mathrm{id}_{K_0(A)}$, which forces M to be the identity matrix. Thus $vu = [A] \in K_0(A^{\mathrm{op}} \otimes A)$, i.e. $vu = \mathrm{id}_A$ in $\mathcal{C}$. $\square$

Corollary 5.2. *For any integer* n, *any central simple F-algebra D, and any F-variety V,*

$$K_n(V; A \otimes_F D) \cong K_n(V \times S; D).$$

In particular, $K_n(F) \oplus K_n(B) \oplus K_n(Q) = K_n(A) \cong K_n(S)$.

Proof. For any central simple F-algebra D, Morita Equivalence gives a natural isomorphism $K_0(S; A^{\mathrm{op}} \otimes_F D^{\mathrm{op}} \otimes_F D) = K_0(S; A^{\mathrm{op}})$. Thus the isomorphism $u :$

$A \to S$ in $\mathcal{C}$ also defines an isomorphism from $A \otimes_F D$ to (S, D) in $\mathcal{C}$. Applying the realization functor K_n^V yields $K_n(V; A \otimes_F D) \cong K_n(V \times S; D)$. $\square$

We conclude the paper with an Index Reduction Formula for the function field of the surface $S(B, Q, KL)$. We will need the following lemma:

Lemma 5.3 ([10]). *Let X be a irreducible F-variety, and D a central simple F-algebra. The restriction homomorphism $K_0(X; D) \to K_0(D_{F(X)})$ induced by the inclusion $\mathrm{Spec}(F(X)) \to X$ is surjective.*

Lemma 5.4. *Let X be an irreducible F-variety, and D a central simple F-algebra.*

$$\mathrm{ind}\, D_{F(X)} = \frac{1}{\deg D} \, \mathrm{g.\,c.\,d.}\{\mathrm{rank}(P), \forall P \in \mathcal{P}(X; D)\}.$$

Proof. We recall that for any field E and any central simple E-algebra D', $K_0(D')$ is cyclic, generated by the class of a simple D'-module M'. Moreover, $\dim_E(M') = \deg(D') \,\mathrm{ind}(D')$.

The rank homomorphism $\mathrm{rank} : K_0(X; D) \to K_0(F(X))$ has the following decomposition:

$$K_0(X; D) \to K_0(D_{F(X)}) \to K_0(F(X)),$$

where the first map is induced by the inclusion $\mathrm{Spec}(F(X)) \to X$, and the second map takes the class of a $D_{F(X)}$-module to the class of the corresponding $F(X)$-vector space.

As $K_0(F(X))$ is cyclic, the image of the rank homomorphism is $n[F(X)]$, where n is the greatest common divisor of the numbers $\mathrm{rank}(P)$, for all $P \in \mathcal{P}(X; D)$. By the previous lemma, the homomorphism $K_0(X; D) \to K_0(D_{F(X)})$ is surjective. Thus if M is a simple $D_{F(X)}$-module,

$$n = \dim_{F(X)}(M) = \deg(D_{F(X)}) \,\mathrm{ind}(D_{F(X)}) = \deg(D) \,\mathrm{ind}(D_{F(X)}),$$

and the result follows. $\square$

Corollary 5.5 (Index Reduction Formula). *Let $S = S(B, Q, KL)$ be a del Pezzo surface of degree 6. For any central simple F-algebra D, $\mathrm{ind}\, D_{F(S)}$ is equal to:*

    i. $\mathrm{g.\,c.\,d.}\{\mathrm{ind}(D), 2\,\mathrm{ind}(D \otimes_F B), 3\,\mathrm{ind}(D \otimes_F Q)\}$*, if K and L are fields.*

    ii. $\mathrm{g.\,c.\,d.}\{\mathrm{ind}(D), \mathrm{ind}(D \otimes_F B_1), \mathrm{ind}(D \otimes_F B_2)\}$*, if $K = F \times F$ and L is a field. Here $B = B_1 \times B_2$.*

    iii. $\mathrm{g.\,c.\,d.}\{\mathrm{ind}(D), \mathrm{ind}(D \otimes_F Q_1), 2\,\mathrm{ind}(D \otimes_F Q_2)\}$*, if K is a field, and $L = F \times E$. Here $Q = Q_1 \times Q_2$.*

    iv. $\mathrm{g.\,c.\,d.}\{\mathrm{ind}(D), \mathrm{ind}(D \otimes_F Q_1), \mathrm{ind}(D \otimes_F Q_2), \mathrm{ind}(D \otimes_F Q_3)\}$*, if K is a field, and $L = F \times F \times F$. Here $Q = Q_1 \times Q_2 \times Q_3$.*

    v. $\mathrm{ind}\, D$*, when K and L are not fields.*

Remark 5.6. In case ii., $Q = M_2(L)$ is necessarily split, as K is not a field. Then $\mathrm{ind}(D \otimes_F M_2(L)) = \mathrm{ind}(D_L)$, and as $\mathrm{ind}(D)$ divides $[L : F]\,\mathrm{ind}(D_L) = 3\,\mathrm{ind}(D_L)$, the greatest common divisor will not change if we remove the term

$3 \operatorname{ind}(D \otimes_F Q)$. Similarly, in cases iii., iv., and v., we can remove the term with the split B or Q when computing greatest common divisors.

Proof. As $u : A \to S$ is an isomorphism in $\mathcal{C}$, it defines an isomorphism $K_0(u)$ from $K_0(A \otimes_F D)$ to $K_0(S; D)$. Moreover, as $A = F \times B \times Q$, $K_0(A \otimes_F D) \cong K_0(D) \oplus K_0(B \otimes_F D) \oplus K_0(Q \otimes_F D)$.

We will consider the case where K and L are fields. The proof of the other cases are similar. As D, $B \otimes_F D$, and $Q \otimes_F D$ are central simple algebras (with centers F, K, and L, respectively), their K_0 groups are cyclic, generated by the class of a simple module. Therefore by Lemma 5.4, $\deg(D) \operatorname{ind}(D_{F(S)})$ will equal the greatest common divisor of the ranks of the images of simple D, $B \otimes_F D$ and $Q \otimes_F D$ modules under the image of $K_0(u) : K_0(A \otimes_F D) \to K_0(S; D)$.

So let M_B be a simple $B \otimes_F D$-module. Then $\dim_K(M_B) = \deg(B \otimes_F D) \operatorname{ind}(B \otimes_F D)$, and thus

$$
\begin{aligned}
\operatorname{rank}(K_0(u)(M_B)) &= \operatorname{rank}(M_B \otimes_B \mathcal{I}) \\
&= \frac{\dim_F(M_B) \operatorname{rank}(\mathcal{I})}{\dim_F(B)} \\
&= \frac{\dim_K(M_B) \operatorname{rank}(\mathcal{I})}{\dim_K(B)} \\
&= \frac{\deg(B \otimes_F D) \operatorname{ind}(B \otimes_F D) \operatorname{rank}(\mathcal{I})}{\dim_K(B)} \\
&= 2 \deg(D) \operatorname{ind}(B \otimes_F D).
\end{aligned}
$$

Similarly, if M_Q (resp. M_F) is a simple $Q \otimes_F D$-module (resp. D-module), $\operatorname{rank}(K_0(u)(M_Q)) = 3 \deg(D) \operatorname{ind}(D \otimes_F Q)$ (resp. $\operatorname{rank}(K_0(u)(M_F)) = \deg(D) \operatorname{ind}(D))$, and the result follows. $\qquad\square$

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